JOURNAL OF DIFFERENTIAL EQUATIONS 4, 634-645 (1968)

A Boundary Value Problem for an Equation of Mixed Type Having Two Transitions*

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Received December 12, 1966

The Tricomi problem for the generalized Tricomi equation

$$Lu = K(z)\, u_{xx} + u_{zz} = f, \tag{1}$$

where $K(z)$ is a monotone increasing function of z which vanishes for $z = 0$, has been treated previously by various authors (see [*1*], [*2*], [*3*], [*4*], [*5*]).

The object of this paper is to examine a more general class of mixed equations which occur, for example, in plane compressible flows with an aligned magnetic field. In this case, unlike ordinary compressible flow, there exists the possibility of three types of transition from elliptic to hyperbolic. The question then arises: Does there exist a two-dimensional channel flow in which all three changes take place? To solve this problem, one first tries to find related boundary value problems which are properly posed in the hodograph plane. The simplest of these is an extension of the Tricomi problem described here. For a detailed description of this physical situation see Seebass [*6*].

We shall consider equations of the type (1) where $K(z)$ is monotone increasing for negative z, monotone decreasing for positive z, and $K(1) = K(-1) = 0$. We shall assume that $K(z)$ has continuous third derivatives and that $K'(-1) > 0$ and $K'(-1) < 0$. The simplest example of such a function K is the function $K(z) = 1 - z^2$.

We shall solve a boundary value problem for this equation in the domain $G = G_1 \cup G_2 \cup G_3$ shown in Fig. 1, where $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are characteristics of the operator L, and γ_1 and γ_2 are smooth curves which coincide with the "normal curves" of Tricomi (see [*1*]) in some small neighborhood of the points A_+, B_+, A_-, B_-. The function $u(x, z)$ will be called a solution of the Tricomi problem in G if $Lu = f$ in G, with f Hölder continuous of

* Prepared under the auspices of the National Science Foundation grant GP-4079.

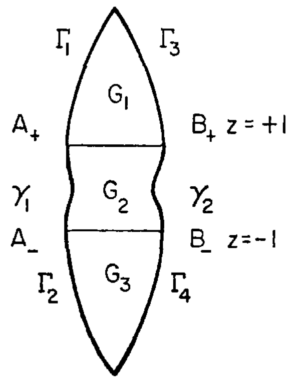

FIG. 1

some order, u vanishes on Γ_1, Γ_2, γ_1, and γ_2, and u and its first partial derivatives are Hölder continuous in the interior of G.

The main result is a Fredholm alternative which we state as

THEOREM 1. *Either there exists a unique solution to the Tricomi problem in G or the homogeneous problem has non-trivial solutions.*

It is worth noting that under certain hypotheses on the function $K(z)$ uniqueness holds. It follows from the maximum principle for elliptic equations that any solution of the homogeneous equation in G_2 which vanishes on γ_1 and γ_2 assumes its maximum on one of the parabolic straight lines and, at that point, its outward pointing normal derivative is greater than or equal to zero. Assuming that $K(z)$ satisfies the conditions of the Agmon, Nirenberg, Protter maximum principle (see [7]) in the two hyperbolic domains, we know, on the other hand, that any solution of the hyperbolic problem which vanishes on a characteristic assumes its maximum on the parabolic line and has there a strictly positive normal derivative with respect to the hyperbolic domain. It follows that, under these assumptions on $K(z)$, the homogeneous problem has only the trivial solution. For $K(z) = 1 - z^2$ the maximum principle holds in G_1 and G_3 if the vertical height of these characteristic triangles is not too large.

I

Let $\mathscr{C}^\alpha(G)$ be the Banach space of functions defined on G which are Hölder continuous with exponent α, $0 < \alpha < 1$. Let $\mathscr{C}^{1+\alpha}(G)$ be the Banach space of functions which together with their first partial derivatives belong to $\mathscr{C}^\alpha(G)$.

We describe the method of proof of Theorem 1. Assuming there is a unique solution $u(x, z)$ of the boundary value problem described above, denote its value on the x-axis by $\varphi(x) = u(x, 0)$. The object is to obtain a functional relation for φ in the form of a Fredholm alternative.

We set $G_2 = G_2^+ \cup G_2^-$ where G_2^+ is the part of G_2 lying above the x-axis and G_2^- the part below. Using the results of Gellerstedt and Protter (see [*2*] and [*3*]) one may then determine solutions $u_+(x, z)$ and $u_-(x, z)$ of the two boundary value problems corresponding to the two domains $D_+ = G_2^+ \cup G_1$ and $D_- = G_2^- \cup G_3$.[1] These solutions will, of course, depend upon the function $\varphi(x)$. In fact, using the techniques of potential theory, one obtains a representation for u_+ and u_- in the domains D_+ and D_-, respectively.

Since the representation described above becomes singular on the x-axis, it is not reasonable to attempt to solve an integral equation for φ there. Instead, we consider the values of u_+ and u_- on two intermediate lines $z = \pm c$, where $0 < | c | < 1$. On these lines

$$\begin{aligned} u_+(x, c) &= T_+\varphi + f_+(x, c) \\ u_-(x, -c) &= T_-\varphi + f_-(x, -c), \end{aligned} \tag{A}$$

where T_+ and T_- are certain integral operators and f_+ and f_- are inhomogeneous terms not depending on φ.

Let M be the domain bounded by the straight lines $z = \pm c$ and the curves γ_1 and γ_2. In M, $u(x, z)$ satisfies the ***uniformly elliptic*** equation, $Lu = f$ and, on the boundary of M, we have

$$u = \begin{cases} 0 & \text{on } \gamma_1 \text{ and } \gamma_2 \\ u_+ & \text{on } z = c \\ u_- & \text{on } z = -c. \end{cases}$$

Set $u(x, z) = \psi + \omega$ where ψ is a solution of the homogeneous equation, ω is a solution of the inhomogeneous equation, and the boundary conditions on ∂M are:

$$\psi = \begin{cases} 0 & \text{on } \gamma_1 \text{ and } \gamma_2 \\ T_+\varphi & \text{on } z = c \\ T_-\varphi & \text{on } z = -c \end{cases}$$

$$\omega = \begin{cases} 0 & \text{on } \gamma_1 \text{ and } \gamma_2 \\ f_+ & \text{on } z = c \\ f_- & \text{on } z = -c. \end{cases}$$

[1] It is necessary here to assume that uniqueness holds for each of these problems.

If we set $\psi(x, 0) = S\varphi$, we obtain the following functional relation on the x-axis:

$$(1 - S)\,\varphi = \omega(x, 0). \tag{2}$$

The operator S is clearly linear. To establish the theorem, we must obtain the representation (A) and then show that S is a compact operator on some sufficiently smooth Hölder class. For this purpose, we prove two lemmas.

LEMMA 1. *If $\varphi \in \mathscr{C}^{1+\alpha}$ for some α, $0 < \alpha < 1$, then, on the lines $z = \pm c$, $u_\pm = T_\pm\varphi + f_\pm$. The functions $T_\pm\varphi, f_\pm$ belong to $\mathscr{C}^{1+\beta}$ for any β, $0 < \beta < 1$, and the operators $T_\pm$ are continuous mappings of $\mathscr{C}^{1+\alpha}$ into $\mathscr{C}^{1+\beta}$, for any β.*

We note that the smoothness properties of $u_\pm$ on the lines $z = \pm c$ are a simple consequence of general elliptic theory. However, in the derivation of (A) these properties will automatically appear.

Consider the boundary value problem $Lv = g$ in M with v given on ∂M. Suppose that g is Hölder continuous in M of some order, and that the boundary values of v belong to $\mathscr{C}^{1+\beta}$ for any β, $0 < \beta < 1$, on each smooth segment of ∂M. Then

LEMMA 2. *There is a unique solution, $v(x, z)$, of the boundary value problem described above, which is twice continuously differentiable in the interior of M, and belongs to $\mathscr{C}^{1+\gamma}$ in the closure of M. The Hölder exponent γ depends only on the shape of M, and the norm of v in $\mathscr{C}^{1+\gamma}$ depends continuously on the boundary data.*

It follows from Lemma 1 and Lemma 2 that the functions $\psi(x, z)$ and $\omega(x, z)$ belong to $\mathscr{C}^{1+\gamma}(\bar{M})$. In particular, on the x-axis $S\varphi = \psi(x, 0)$ belongs to $\mathscr{C}^{1+\gamma}$ for $\varphi \in \mathscr{C}^{1+\alpha}$. Furthermore, S is a continuous mapping of $\mathscr{C}^{1+\alpha}$ into $\mathscr{C}^{1+\gamma}$. Therefore, S is compact on $\mathscr{C}^{1+\alpha}$ for $\alpha < \gamma$, and under the assumption that uniqueness holds we may write

$$\varphi = (1 - S)^{-1}\omega. \tag{3}$$

Since $S\varphi$ and ω are twice continuously differentiable in the interior of M, it follows that φ is twice continuously differentiable except possibly at the endpoints of the interval.

Having obtained the representation (3) for φ, it is now possible to solve the original problem. Knowing φ, one may determine the solutions u_+ and u_- in D_+ and D_-. We claim that the solution u of the Tricomi problem in G is the function

$$u(x, z) = \begin{cases} u_+ & \text{in } D_+ \\ u_- & \text{in } D_- \end{cases}$$

The only point which must be verified is the differentiability of $u(x, z)$ across the x-axis. To demonstrate this, let

$$\hat{u} = \psi + \omega$$

in the domain M, with ψ, ω and M as defined above. (The existence of $\hat{u}$ is a consequence of Lemma 2.)

From the choice of φ we see that $\hat{u}$ is a solution of $Lu = f$ which agrees with u_+ on ∂G_2^+ and with u_- on ∂G_2^-. From the maximum principle $\hat{u} = u_+$ in $\overline{G_2}^+$ and $\hat{u} = u_-$ in $\overline{G_2}^-$. Since $\hat{u}$ has continuous second derivatives across the x-axis, the functions u_+ and u_- must, together with second derivatives, agree across the x-axis.

In fact, the above argument shows that the solution $u(x, z)$ has Hölder continuous first derivatives everywhere in the closure of G with the exception of the parabolic points A_+, A_-, B_+, B_- where it exhibits the characteristic Tricomi singularity; that is, the normal derivative of u may become infinite of some order less than one.

Lemma 2 is a consequence of elliptic theory and shall not be proved here. A more general version of it may be found in [8].

II

We first investigate the Green's function for the Tricomi equation

$$yu_{xx} + u_{yy} = 0 \tag{4}$$

in the domain D' shown in Fig. 2. Let C be the piecewise smooth curve consisting of τ_1, τ_2 and the line $y = y_0$. Assign the counter-clockwise orientation to C and assume C to be given in parametric form by $x = x(t)$, $y = y(t)$, with $B = (x(0), y(0))$, $B' = (x(t_1), y(t_1))$, $A' = (x(t_2), y(t_2))$ and $A = (x(1), y(1))$. C is to coincide with the normal curve in some small neighborhood of the points A and B.

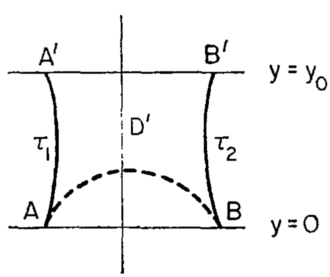

FIG. 2

The Green's function $G(\xi, \eta, x, y)$ for Eq. (4) is defined as a symmetric solution of Eq. (4) in the variables (ξ, η) and (x, y) which has a logarithmic singularity at $\xi = x, \eta = y$ but is otherwise regular in the elliptic half-plane, $y > 0$. For $(x, y) \in D'$, G satisfies the following boundary conditions for (ξ, η) on $\partial D'$:

$$\begin{aligned} G &= 0 \quad \text{on} \quad C \\ \frac{\partial G}{\partial n} &= 0 \quad \text{for} \quad y = 0. \end{aligned} \tag{B}$$

These are the appropriate homogeneous boundary conditions for the Tricomi equation in an elliptic domain containing part of the parabolic line as boundary.

The existence of the Green's function has been demonstrated by Gellerstedt (see [*2*]) for domains in which the curve C has a continuously turning tangent and his proof may be directly extended to the case where C has a finite number of corners.

We shall need to know the following facts shown by Gellerstedt about the Green's function:

$$G(x, 0, \xi, \eta) = 0(|(x - \xi)^2 + \tfrac{4}{9}\eta^3|^{-(1/6)-\epsilon}) \quad \text{for any} \quad \epsilon > 0$$
$$\text{as} \quad \xi \to x, \quad \eta \to 0. \tag{6}$$

$$G(x, y, \xi, \eta) = 0(|\log r|), \quad r = ((x - \xi)^2 + (y - \eta)^2)^{1/2}, \quad y > 0, \tag{7}$$

as $\xi \to x, \eta \to y$.

Define the operator $A_t[\cdot]$ along any curve: $y = y(t)$, $x = x(t)$, by

$$A_t[\cdot] = y \frac{dy}{dt} \frac{\partial}{\partial x} - \frac{dx}{dt} \frac{\partial}{\partial y}.$$

For $(\xi(s), \eta(s))$ and $(x(t), y(t))$ on the curve C,

$$A_t[G(x(t), y(t), \xi(s), \eta(s)] = 0(|\log r|) \quad \text{as} \quad s \to t \tag{8}$$

and has a jump discontinuity at A' and B'.

We remark that along any straight line parallel to the x-axis, $A_t[\cdot]$ represents normal differentiation.

The results of this section and Green's formula enable us to represent any solution of Eq. (4) in D' as

$$\begin{aligned} U(x, y) = &- \int_a^b \frac{\partial U}{\partial n}(\xi, 0)\, G(\xi, 0, x, y)\, d\xi \\ &- \int_0^1 U(\xi(t), \eta(t))\, A_t[G(\xi(t), \eta(t), x, y)]\, dt \end{aligned} \tag{9}$$

where $A = (a, 0)$ and $B = (b, 0)$.

In the future we will refer to the first integral on the right as $I_1(x, y)$ and the second as $I_2(x, y)$.

III

Next, we derive the representation described above for the functions u_+ and u_-.

Let $\gamma_1 = \gamma_1^+ \cup \gamma_1^-$ and $\gamma_2 = \gamma_2^+ \cup \gamma_2^-$ where $+$ denotes the parts of these curves lying above the x-axis and $-$ that below. Suppose that the x-axis intersects γ_1 at $x = a'$ and γ_2 at $x = b'$ and let $I = [a', b']$. Let $\varphi(x)$ belong to $\mathscr{C}^{1+\alpha}(I)$ for some given α and suppose $\varphi(a') = \varphi(b') = 0$.

We seek solutions of the boundary value problems:

$$Lu_\pm = f \quad \text{in} \quad D_\pm$$

with

$$u_\pm(x, 0) = \varphi(x) \qquad a' \leqslant x \leqslant b'$$

$$u_\pm = 0 \quad \text{on} \quad \gamma_1^\pm \quad \text{and} \quad \gamma_2^\pm$$

$$u_\pm = 0 \quad \text{on} \quad \Gamma_1 \text{ (resp., } \Gamma_2).$$

Since each of the above problems is identical, we restrict our attention to u_-. If we set

$$u_-(x, z) = \left(\frac{y}{K(z)}\right)^{1/4} v(x, y)$$

$$\frac{dz}{dy} = \left(\frac{y}{K(z)}\right)^{1/2} \qquad \text{with } y(-1) = 0 \quad \text{and} \quad y(0) = y_0 > 0$$

we obtain a new boundary value problem for the function $v(x, y)$ in the domain $D = D' \cup D''$ shown in Fig. 3, where Γ' and Γ'' are the characteristics of the Tricomi equation and τ_1 and τ_2 are the images of γ_1^- and γ_2^-.

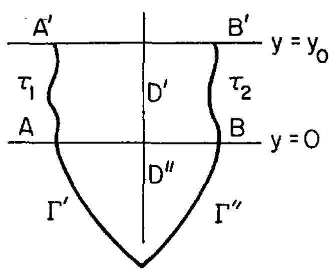

FIG. 3

In D the function $v(x, y)$ is a solution of

$$\hat{L}v = yv_{xx} + v_{yy} - c(y)\, v = F(x, y) \tag{10}$$

where $c(y)$ and $F(x, y)$ are bounded. The boundary conditions for v are:

$$v = 0 \quad \text{on} \quad \tau_1, \tau_2 \quad \text{and} \quad \Gamma'$$

$$v(x, y_0) = c_0\varphi(x) \quad \text{for} \quad a' \leqslant x \leqslant b'$$

where $c_0 = (K(0)/y_0)^{1/4}$.

The existence of the solution $v(x, y)$ has been shown by Protter (see [3]). Therefore, we need only concern ourselves with a formula for $v(x, y)$ in terms of φ on some line, $y =$ const. lying in D'. For this purpose we decompose v as a sum of functions:

$$v(x, y) = U(x, y) + V_1(x, y) + V_2(x, y) + w(x, y).$$

Using (9), where $U(x, y)$ is the solution of the Tricomi equation in D' which agrees with $v(x, y)$ on C and has the same normal derivative on the x-axis,

$$\begin{aligned} U(x, y) = I_1 + I_2 = &-\int_a^b \frac{\partial v}{\partial n}(\xi, 0)\, G(\xi, 0, x, y)\, d\xi \\ &+ \int_{t_1}^{t_2} c_0\varphi(\xi) \frac{\partial}{\partial \eta}(G(\xi, y_0, x, y))\, d\xi. \end{aligned}$$

We remark that $\partial v/\partial n(\xi, 0)$ has been shown, by Protter, to be a solution of a certain Volterra integral equation considered by Tricomi (see [1]). Obviously, $\partial v/\partial n(\xi, 0)$ depends linearly on φ, and the results of Tricomi show that if $\varphi \in \mathscr{C}^{1+\alpha}$, then $\partial v/\partial n(\xi, 0)$ is a continuous function except at A and B where it becomes infinite of some order less than one.

The functions w, V_1, and V_2 satisfy the homogeneous boundary condition (B) and they are solutions, respectively, of

$$\hat{L}w = F(x, y)$$

$$\hat{L}V_1 = c(y)\, I_1$$

$$\hat{L}V_2 = c(y)\, I_2\,.$$

We shall make use of the following remarks concerning solutions $\mu(x, y)$ of Eq. (10) in D' which satisfy the boundary conditions (B):

(a) If $\hat{L}\mu = 0$, then $\mu \equiv 0$.

(b) $\hat{L}\mu = g$ implies that μ is a solution of the integral equation

$$\mu(x, y) + \iint_{D'} c(\eta)\, G(\xi, \eta, x, y)\, \mu(\xi, \eta)\, d\xi\, d\eta = \iint_{D'} g(\xi, \eta)\, G(\xi, \eta, x, y)\, d\xi\, d\eta \tag{11}$$

where G is the Green's function for the Tricomi equation in D'. (Remark (a) follows from a theorem of Agmon, Nirenberg, and Protter in [1].) Rewriting (11) as

$$(1 + T)\mu = k(x, y) \tag{11'}$$

it is clear from the behavior of the Green's function that the integral operator T behaves exactly as a logarithmic potential in any subdomain D'' of D' which does not include the x-axis. From (a) the homogeneous problem has only the trivial solution. Therefore, on any line in D', say, $y = y_1$, with $0 < y_1 < y_0$, we have a unique solution of (11') belonging to $\mathscr{C}^{1-\beta}$ for any β, $0 < \beta < 1$.

We next determine the functions V_1 and V_2 as potentials on some intermediate line, $y = y_1$, $0 < y_1 < y_0$. For this purpose we seek functions $m(t, x, y)$ and $n(t, x, y)$ such that

$$V_1(x, y) = \int_a^b \frac{\partial v}{\partial n}(t, 0)\, n(t, x, y)\, dt$$

$$V_2(x, y) = \int_{t_1}^{t_2} c_0 q(t)\, m(t, x, y)\, dt.$$

Substituting these expressions into (11), one sees that m and n must be solutions of:

$$(1 + T)\, n(t, x, y) = \iint_{D'} c(\eta)\, G(t, 0, \xi, \eta)\, G(\xi, \eta, x, y)\, d\xi\, d\eta = g_1(t, x, y) \tag{12}$$

$$(1 + T)\, m(t, x, y) = \iint_{D'} c(\eta) \frac{\partial G}{\partial y}(t, y_0, \xi, \eta)\, G(\xi, \eta, x, y)\, d\xi\, d\eta = g_2(t, x, y) \tag{13}$$

with T as defined in (11').

For fixed t, T is a compact operator on $\mathscr{C}^{1+\beta}(D'')$. Furthermore, if m and n depend continuously on t, then Tm and Tn depend continuously on t. We shall show

(i) $g_1(t, x, y)$ and $g_2(t, x, y)$ are Hölder continuously differentiable in x and y for $0 < y < y_0$, with any exponent β, $0 < \beta < 1$.

(ii) $g_1(t, x, y)$ and $g_2(t, x, y)$ are continuous in t if $0 < y < y_0$.

For the moment let us assume (i) and (ii) hold. Making use of (a) and the above remarks, we may solve a Fredholm alternative for m and n. On any line $y = y_1$ in D', $0 < y_1 < y_0$, $m(t, x, y)$ and $n(t, x, y)$ are continuous functions of t and Hölder continuously differentiable in x of all orders, Proof of (ii). Fix the point (x, y) in D'. Let $D' = D_1' \cup D_2'$, where $\eta \leqslant \frac{1}{2}y$ in D_1' and $\eta > \frac{1}{2}y$ in D_2'. Setting

$$g_1(t, x, y) = \iint_{D_1'} + \iint_{D_2'} c(\eta)\, G(t, 0, \xi, \eta)\, G(\xi, \eta, x, y)\, d\xi\, d\eta$$

it is clear that the second integral in this expression is a continuous function of t. Let

$$r^2 = (t - \xi)^2 + \eta^2$$

$$C_\delta = \{(\xi, \eta) | (t - \xi)^2 + \eta^2 < \delta\}, \text{ for some small positive } \delta.$$

There is a $\mathscr{C}_\infty$ function $\chi_\delta(r)$, $0 \leqslant \chi \leqslant 1$, which vanishes for $r \leqslant \delta/2$ and is equal to one for $r \geqslant \delta$.

Consider the sequence of functions

$$\nu_\delta(t, x, y) = \iint_{D_1'} c(\eta)\, G(\xi, \eta, x, y)\, \chi_\delta G(t, 0, \xi, \eta)\, d\xi\, d\eta\,.$$

Clearly, for fixed (x, y) not belonging to D_1', ν_δ is a continuous function of t. Furthermore, ν_δ converges uniformly in t as δ tends to zero. This may be seen as follows:

$$\left| \nu_\delta - \iint_{D_1'} c(\eta)\, G(\xi, \eta, x, y)\, G(t, 0, \xi, \eta)\, d\xi\, d\eta \right| \leqslant K \iint_{C_\delta} | G(t, 0, \xi, \eta)\, d\xi\, d\eta.$$

We now make use of the estimate (6) for G. Since

$$|(t - \xi)^2 + \eta^2|^{3/2} \leqslant c\, |(t - \xi)^2 + \tfrac{4}{9}\eta^3|,$$

where c is some constant, it follows that for $(\xi, \eta) \in \mathscr{C}_\delta$

$$| G(t, 0, \xi, \eta)| \leqslant \text{const } \delta^{-1/2-\epsilon}, \qquad \text{for any} \quad \epsilon > 0.$$

Therefore, the integral on the right tends to zero uniformly in t. It follows that the limit is continuous in t.

To prove this statement for $g_2(t, x, y)$ we make use of the estimate (8) for $A_t[G]$. This completes the proof of (ii).

Proof of (i). We now fix the point t on the x-axis. Set $D' = D_1' \cup D_2'$ where $\eta \leqslant \eta'$ in D_1' and $\eta > \eta'$ in D_2'. We assume, as well, that $y \geqslant 2\eta'$. Set

$$g_1(t, x, y) = \iint_{D_1'} + \iint_{D_2'} c(\eta)\, G(t, 0, \xi, \eta)\, G(\xi, \eta, x, y)\, d\xi\, d\eta.$$

In D_1' define as before the sequence of functions $\nu_\delta(t, x, y)$. Since $(\xi, \eta) \in D_1'$ and $(x, y) \in D_2'$, these functions are Hölder continuously differentiable in x and y with any exponent less than one. In fact, it follows from (6) that they converge in $\mathscr{C}^{1+\alpha}(D_1')$. Therefore, the first integral in this expression is Hölder continuously differentiable in x and y with any Hölder exponent.

For $(\xi, \eta) \in D_2'$ the function $G(t, 0, \xi, \eta)$ is a regular function. Hence the second integral behaves exactly like a logarithmic potential. From the results of classical potential theory, we know that this integral is Hölder continuously differentiable in x and y with any Hölder exponent less than one. The same procedure proves (i) for the function $g_2(t, x, y)$ where, as before, we use the estimate (8) instead of (6). This completes the proof of (i).

Finally, we may express the desired solution $v(x, y)$ of (10) as:

$$v(x, y) = \int_a^b \frac{\partial v}{\partial n}(\xi, 0)(n(\xi, x, y) - G(\xi, 0, x, y))\, d\xi$$
$$+ \int_{t_1}^{t_2} c_0\varphi(\xi)(m(\xi, x, y) + \frac{\partial G}{\partial \eta}(\xi, y_0, x, y))\, d\xi + w(x, y). \quad (14)$$

More briefly,

$$v(x, y) = w(x, y) + \int_a^b \frac{\partial v}{\partial n}(\xi, 0)\, M(\xi, x, y)\, d\xi + \int_{t_1}^{t_2} \varphi(\xi)\, N(\xi, x, y)\, d\xi. \quad (14')$$

From the preceding discussion, M, N, and w are Hölder continuously differentiable functions in x and y with any exponent $\beta < 1$, provided $0 < y < y_0$. As functions of ξ, M and N are continuous.

Returning to the original problem for the function $u_-(x, z)$ we have

$$u_-(x, -c) = k_c w(x, y(-c))$$
$$+ k_c \left(\int_a^b \frac{\partial v}{\partial n}(\xi, 0)\, M(\xi, x, y(-c))\, d\xi + \int_{t_1}^{t_2} \varphi(\xi)\, N(\xi, x, y(-c))\, d\xi\right) \quad (15)$$

where $0 < y(-c) < y_0$ and $k_c = (y(-c)/K(-c))^{1/4}$. Denoting the sum of the two integrals by $T_-\varphi$ and setting $f_- = k_c w(x, y(-c))$ gives

$$u_- = T_-\varphi + f_- .$$

Since a similar formula holds for u_+, this completes the proof of the lemma and of Theorem 1.

Acknowledgments

I would like to express my thanks to C. S. Morawetz for suggesting this line of research and to M. Protter for several very helpful conversations.

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